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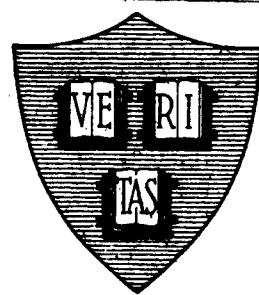
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SCATTERING OF UNIDIRECTIONAL SURFACE WAVES

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By

S. R. Seshadri

February 15, 1963

Technical Report No. 402

Cruft Laboratory  
Harvard University  
Cambridge, Massachusetts

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Abstract

↓ A perfectly conducting plane screen embedded in a gyrotropic medium is shown to be able to support a unidirectional surface wave. Such a surface wave is assumed to be incident on the top of a semi-infinite screen. At the edge, the incident power is converted partly into a reflected surface wave which travels on the bottom of the screen and partly into a space wave. The angular distribution of the radiated energy as well as the power-reflection and the power-transmission coefficients are evaluated. Total reflection is shown to occur for a certain band of frequencies.

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In recent times, interest has arisen in the study of scattering of electromagnetic waves by obstacles embedded in anisotropic media. Wave propagation in a homogeneous, anisotropic medium is more complicated than in a homogeneous, isotropic space since the characteristics of an anisotropic medium, as the name itself implies, are different in different directions. However, in general, there are two categories of problems of scattering in anisotropic media, which are quite similar to those in isotropic space. The scattering by cylindrical obstacles in a uniaxially anisotropic medium constitutes the first category and Felsen [1] is currently carrying out a systematic investigation of these problems. To the second category belong certain two-dimensional problems of scattering by cylindrical obstacles in a gyrotropic medium for the case in which the gyrotropic axis is parallel to the generators of the cylinder and perpendicular to the direction of the incident wave. In this paper, a simple problem belonging to the second category is investigated.

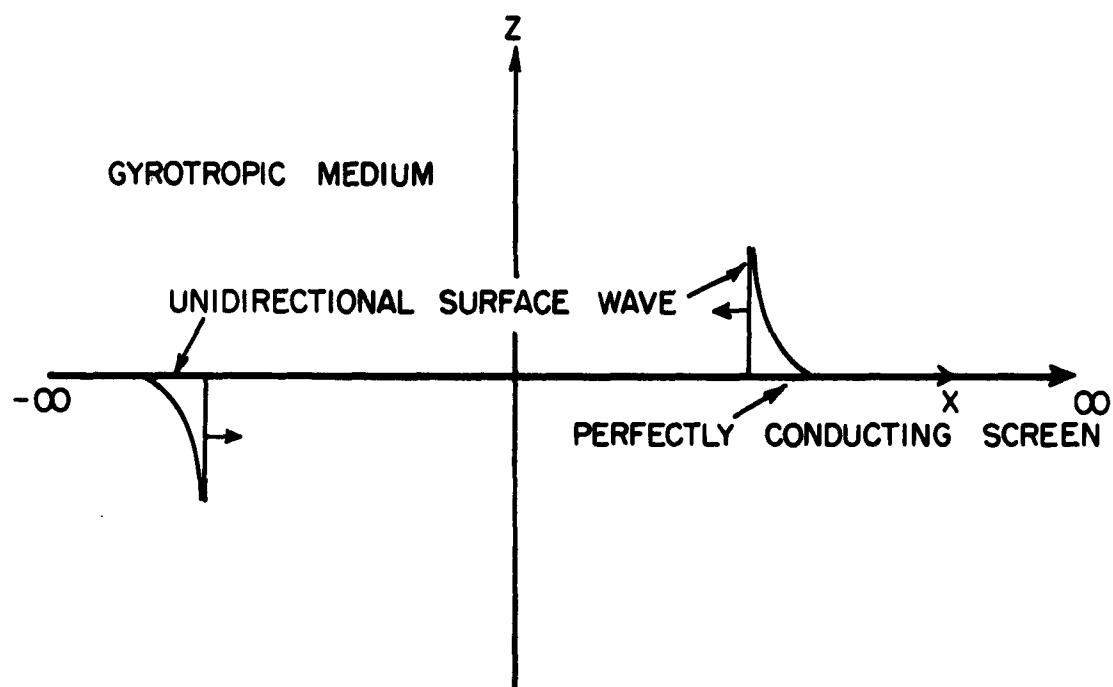
Consider a perfectly conducting screen of infinite extent embedded in a gyrotropic medium. A unidirectional surface wave has been shown [2], [3] to be supported along the screen. This surface wave is a plane TEM wave having its magnetic vector parallel, and its electric vector perpendicular, to the surface of the screen. The external static magnetic field is parallel to the direction of the magnetic vector of the surface wave. For a given sense of the external magnetic field, the surface wave travels only in one direction on the top of the screen and in the opposite direction on the bottom. The

directions of propagation of the surface waves on the top and the bottom of the screen are both reversed, when the sense of the external magnetic field is changed.

Such a unidirectional surface wave is assumed to be incident on the top of a perfectly conducting semi-infinite screen embedded in a gyrotropic medium such that the gyrotropic axis is parallel to the edge of the screen. The unidirectional surface wave is scattered at the open end. Part of the power in the incident wave is carried by the surface wave which travels along the bottom of the screen in a direction opposite to that of the incident surface wave. The remainder of the incident power is carried by the space wave which is excited by the discontinuity. This problem is formulated in terms of a Wiener-Hopf integral equation, which is solved by the well-known function-theoretic methods. Explicit expressions for the reflection and the transmission coefficients, which give the proportion of the incident power carried respectively by the reflected surface wave and the transmitted space wave, are determined. For certain frequency ranges, the entire power in the incident surface wave is carried by the reflected surface wave.

#### Unidirectional Surface Waves

Consider a perfectly conducting screen of infinite extent occupying the region  $-\infty \leq x \leq \infty$ ,  $-\infty \leq y \leq \infty$ , and  $z = 0$ , where  $x$ ,  $y$ ,  $z$  form a right-handed rectangular coordinate system. (Fig. 1). The entire space exterior to the screen is filled with a uniform plasma. A uniform magnetic



$$\begin{aligned}\vec{B} &= \hat{y} B_0 \quad \text{FOR } 0 < \Omega < R \\ &= -\hat{y} B \quad \text{FOR } R < \Omega < \infty\end{aligned}$$

FIG. 1 UNIDIRECTIONAL SURFACE WAVE ALONG AN INFINITE SCREEN

field  $B_0$  is impressed in the positive  $y$  direction throughout the plasma. Under certain simplifying assumptions [2] which are usually made, it is found that in the plasma region, after the usual linearization, the electric and magnetic fields satisfy the time-harmonic Maxwell's equations

$$\nabla \times \vec{E} = i\omega\mu_0 \vec{H} \quad (1)$$

$$\nabla \times \vec{H} = -i\omega \epsilon_0 \vec{\epsilon} \cdot \vec{E} \quad (2)$$

where  $\mu_0$  and  $\epsilon_0$  are the permeability and dielectric constant pertaining to vacuum. A harmonic time dependence  $e^{-i\omega t}$  is assumed for all the field components. The components of the relative dyadic dielectric constant  $\vec{\epsilon}$  are given by the following matrix

$$\vec{\epsilon} = \begin{bmatrix} \epsilon_1 & 0 & -i\epsilon_2 \\ 0 & \epsilon_3 & 0 \\ i\epsilon_2 & 0 & \epsilon_1 \end{bmatrix} \quad (3)$$

where

$$\epsilon_1 = \frac{\Omega^2 - R^2 - 1}{\Omega^2 - R^2}$$

$$\epsilon_2 = \frac{R}{\Omega(\Omega^2 - R^2)}$$

$$\epsilon_3 = 1 - \frac{1}{\Omega^2}$$

$$\Omega = \frac{\omega}{\omega_p} \quad \text{and} \quad R = \frac{\omega_c}{\omega_p} \quad (4)$$

The plasma frequency  $\omega_p$  and the gyromagnetic frequency  $\omega_c$  of the electrons are given by

$$\omega_c = -\frac{eB_0}{m}, \quad \omega_p^2 = \frac{Ne^2}{m\epsilon_0} \quad (5)$$

where  $e$  is the charge of the electron,  $m$  is the mass of the electron and  $N$  is the average density of electrons.

Only the two-dimensional problem for which all the field components are independent of the  $y$  coordinate will be considered. For this case, the electromagnetic field is separable into  $E$  and  $H$  modes. Since unidirectional surface waves are present only in the case of the  $E$  mode, the  $H$  mode will not be considered. For the  $E$  mode, only a single component of the magnetic field, namely,  $H_y$  is present. With the help of (2), the nonvanishing components of the electric field can be shown to be given by

$$E_x(x, z) = -\frac{i\epsilon_1}{\omega\epsilon_0\epsilon} \frac{\partial}{\partial z} H_y(x, z) - \frac{\epsilon_2}{\omega\epsilon_0\epsilon} \frac{\partial}{\partial x} H_y(x, z)$$

$$E_z(x, z) = \frac{i\epsilon_1}{\omega\epsilon_0\epsilon} \frac{\partial}{\partial x} H_y(x, z) - \frac{\epsilon_2}{\omega\epsilon_0\epsilon} \frac{\partial}{\partial z} H_y(x, z), \quad (6)$$

where

$$\epsilon = \epsilon_1^2 - \epsilon_2^2. \quad (7)$$

With the help of (1) and (6), it is found that  $H_y(x, z)$  satisfies the following wave equation:



$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2 \right] H_y(x, z) = 0 \quad (8)$$

where

$$k^2 = \frac{\omega^2 \mu_0 \epsilon_0 \epsilon}{\epsilon_1} = \frac{k_0^2 \epsilon}{\epsilon_1} \quad (9)$$

In (9),  $k_0$  is the wave number corresponding to vacuum.

In view of (8), it is reasonable to assume the following solution for  $H_y(x, z)$ :

$$H_y^i(x, z) = H_s e^{ik_x x + i\sqrt{k^2 - k_x^2} |z|} \quad (10)$$

where

$$\begin{aligned} \sqrt{k^2 - k_x^2} &= +\sqrt{k^2 - k_x^2} \quad \text{if } k > k_x \\ &= +i\sqrt{k_x^2 - k^2} \quad \text{if } k < k_x \end{aligned} \quad (11)$$

Since the screen is perfectly conducting, the following boundary condition has to be satisfied on the surface of the screen:

$$E_x(x, 0_{\pm}) = 0 \quad (12)$$

For  $z > 0$ , (10) satisfies the boundary condition (12) provided

$$\epsilon_1 \sqrt{k^2 - k_x^2} = i\epsilon_2 k_x \quad (13)$$

With the help of (7) and (9), the solution of (13) for  $k_x$  is easily shown to be given by

$$k_x = k_o \sqrt{\epsilon_1} \quad \epsilon_2 > 0 \quad (14a)$$

$$k_x = -k_o \sqrt{\epsilon_1} \quad \epsilon_2 < 0 \quad (14b)$$

Hence, for  $\epsilon_2 < 0$ , (10) becomes

$$H_y^i(x, z) = H_s e^{-ik_o \sqrt{\epsilon_1} x - \frac{k_o |\epsilon_2| z}{\sqrt{\epsilon_1}}} \quad \text{for } z > 0 \quad (15)$$

In a similar manner for  $z < 0$ , (10) satisfies the boundary condition (12) provided

$$\epsilon_1 \sqrt{k^2 - k_x^2} = -i \epsilon_2 k_x \quad (16)$$

and hence for  $z < 0$

$$k_x = -k_o \sqrt{\epsilon_1} \quad \epsilon_2 > 0 \quad (17a)$$

$$k_x = k_o \sqrt{\epsilon_1} \quad \epsilon_2 < 0 \quad (17b)$$

Therefore, for  $\epsilon_2 < 0$ , (10) becomes

$$H_y^i(x, z) = H_s e^{ik_o \sqrt{\epsilon_1} x - \frac{k_o |\epsilon_2| z}{\sqrt{\epsilon_1}}} \quad \text{for } z < 0 \quad (18)$$

It is clear that (15) and (16) represent surface waves for the range of  $\Omega$  for which  $\epsilon_1 > 0$ . In Fig. 2, a plot of  $\epsilon_1$  as a function of  $\Omega$  is given. An examination of Fig. 2 shows that  $\epsilon_1 > 0$  in the following frequency ranges:

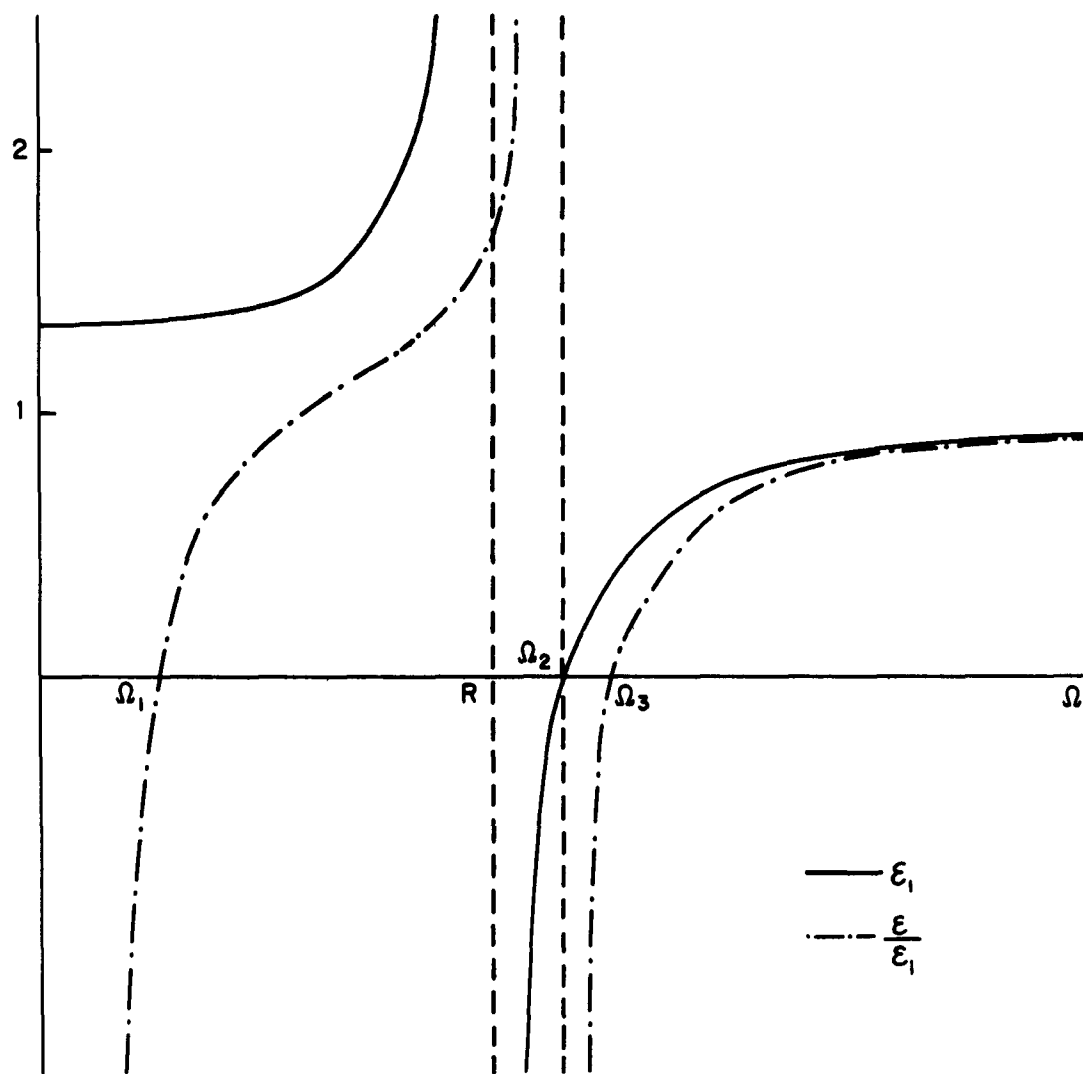


FIG. 2 PLOT OF  $\epsilon_1$  AND  $\frac{\epsilon}{\epsilon_1}$  AS A FUNCTION OF  $\Omega$

$$0 < \Omega < R \text{ and } \sqrt{1+R^2} < \Omega < \infty .$$

Note that  $R$  is positive. From the expression of  $\varepsilon_2$  given in (4), it is obvious that  $\varepsilon_2 < 0$  for the frequency range  $0 < \Omega < R$  and  $\varepsilon_2 > 0$  for  $\sqrt{1+R^2} < \Omega < \infty$ . It is clear that the sign of  $\varepsilon_2$  will change, if the sense of the external magnetic field is changed. It is assumed in what follows that the external magnetic field is in the positive  $y$  direction for the frequency range  $0 < \Omega < R$  and in the negative  $y$  direction for  $\sqrt{1+R^2} < \Omega < \infty$ . As a consequence  $\varepsilon_2$  is always negative. Therefore, (15) and (18) give  $H_y^i(x, z)$  for  $z > 0$  and  $z < 0$  respectively. For the frequency ranges  $0 < \Omega < R$  and  $\sqrt{1+R^2} < \Omega < \infty$ , (15) and (18) represent surface waves. On the top ( $z > 0$ ) of the screen, the surface wave travels in the negative  $x$  direction and on the bottom ( $z < 0$ ), it travels in the positive  $x$  direction. For the specified orientation of the external magnetic field, a surface wave which travels in the positive  $x$  direction on the top of the screen and in the negative  $x$  direction on the bottom is not obtainable. The surface waves are therefore unidirectional in character. Since for the surface wave,  $E_x(x, z) = 0$ , it is clear that it is a TEM wave with its magnetic and electric vectors respectively parallel and perpendicular to the surface of the screen. The excitation of these surface waves was discussed in an earlier paper [2].

### Scattering of the Surface Wave at the Open-end

It is now desired to examine the effect of terminating the perfectly conducting screen ( $0 \leq x \leq \infty$ ) at  $x = 0$ , on the surface wave given by (15) when it is incident on the top ( $z > 0$ ) from  $x = \infty$  (Fig. 3). It is emphasized that a surface wave traveling in the negative  $x$  direction cannot be sustained on the bottom of the screen. Hence, the incident surface wave is only on the top of the screen. No surface wave can be supported in the region  $x < 0$ ; also a surface wave traveling in the positive  $x$  direction cannot be supported on the top of the screen. Hence, the incident surface wave will be partly reflected back as a surface wave on the bottom and partly converted into a radiation field.

Since only the  $y$  component of the magnetic field is present, the current  $I(x)$  induced on the screen is in the  $x$  direction. When the current term is added to the right-hand side of (2), it follows that  $H_y(x, z)$  satisfies the following inhomogeneous wave equation:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) H_y(x, z) = - \left[ i \frac{\epsilon_2}{\epsilon_1} \frac{\partial}{\partial x} + \frac{\partial}{\partial z} \right] I(x) \delta(z) \quad (20)$$

The solution of (20) is easily obtained to be

$$H_y(x, z) = \frac{i}{4} \left( i \frac{\epsilon_2}{\epsilon_1} \frac{\partial}{\partial x} + \frac{\partial}{\partial z} \right) \int_0^{\infty} I(x') H_0^{(1)} [k \sqrt{(x-x')^2 + z^2}] dx' \quad (21)$$

Together with (6) and (8), it can be shown that for  $z = 0$

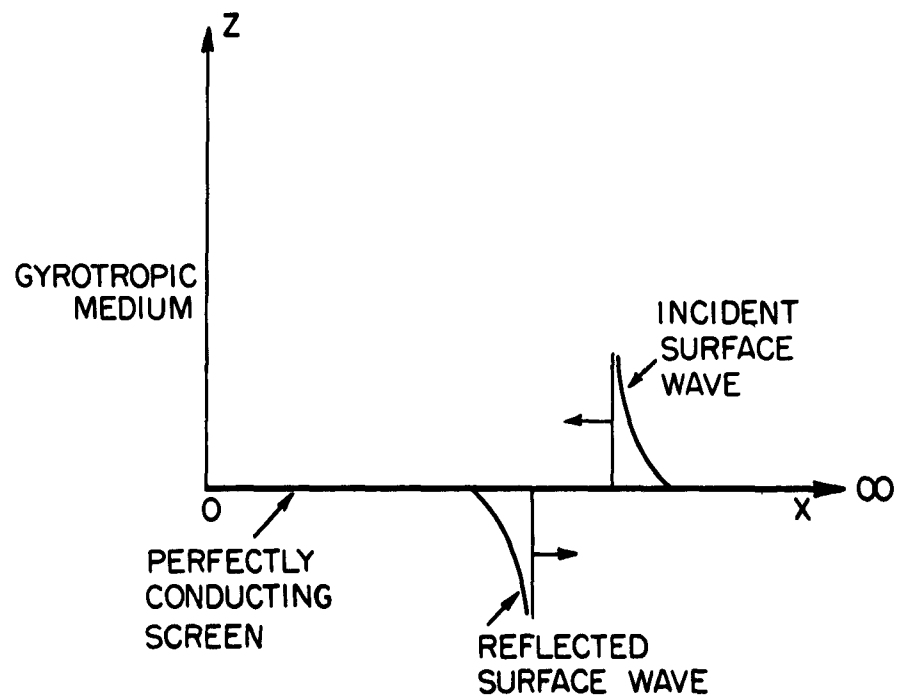


FIG. 3 UNIDIRECTIONAL SURFACE WAVE ALONG A SEMI-INFINITE SCREEN

$$E_x(x, 0) = - \frac{1}{4\omega \epsilon_0 \epsilon_1} \left( \frac{\partial^2}{\partial x^2} + k_0^2 \epsilon_1 \right) \int_0^\infty I(x') H_0^{(1)}[k|x-x'|] dx' \quad (22)$$

Since the perfectly conducting screen occupies only the region  $x > 0$ , the boundary condition (12) holds only for  $x > 0$ .

The application of a Fourier transformation to both sides of (22) yields

$$E_x(\zeta) = - \frac{1}{2\omega \epsilon_0 \epsilon_1} \frac{(k_0^2 \epsilon_1 - \zeta^2)}{\sqrt{k^2 - \zeta^2}} \Gamma(\zeta), \quad (23)$$

where

$$E_x(\zeta) = \int_{-\infty}^0 E_x(x, 0) e^{-i\zeta x} dx \quad (24a)$$

$$\Gamma(\zeta) = \int_0^\infty I(x) e^{-i\zeta x} dx \quad (24b)$$

The branch cuts are defined as in (11).

It is first necessary to know the regions of regularity of the various transforms in (23). From (15) the incident current density is obtained to be

$$I^i(x) = -H_g e^{-ik_0 \sqrt{\epsilon_1} x} \quad (25)$$

Also, as  $x \rightarrow \infty$ ,  $I(x)$  should be of the form

$$I(x) = -H_0 \left[ e^{-ik_0 \sqrt{\epsilon_1} x} + \Gamma e^{ik_0 \sqrt{\epsilon_1} x} \right] \quad (26)$$

where the first term is the incident current density and  $\Gamma$  is the reflection coefficient at  $x = 0$ . It is found from (26) that  $[k_0^2 \epsilon_1 - \xi^2] \Gamma(\xi)$  is regular in the lower half-plane  $\text{Im} \xi < -\text{Im} k$ . Also  $E_x(\xi)$  is regular in the upper half-plane  $\text{Im} \xi > \text{Im} k$ . In addition, the transform of the Hankel function,  $\frac{2}{\sqrt{k^2 - \xi^2}}$  is regular in the strip  $|\text{Im} \xi| < \text{Im} k$ . The transform relation (23) is valid in the strip  $|\text{Im} \xi| < \text{Im} k$  and, therefore, the Wiener-Hopf procedure can be applied to solve (23). It is assumed that  $k$  is real with a small imaginary part which is introduced for convenience and set equal to zero in the final formulas. Rewriting (23) as

$$\frac{[k_0^2 \epsilon_1 - \xi^2] \Gamma(\xi)}{\sqrt{k - \xi}} = -2\omega \epsilon_0 \epsilon_1 \sqrt{k + \xi} E_x(\xi), \quad (27)$$

it is seen that the right-hand side of (27) is regular in the upper half-plane and the left-hand side is regular in the lower half-plane. Both are regular in the strip  $|\text{Im} \xi| < \text{Im} k$  and may be considered as analytic continuations of each other; together they define an integral function. Since the current is perpendicular to the edge, the Meixner Corner Condition requires that  $I(x)$  vanish at  $x = 0$  as  $x^{1/2}$ . Therefore, it follows that  $\Gamma(\xi) \sim \xi^{-3/2}$  as  $\xi \rightarrow \infty$  in the lower half-plane. The asymptotic behavior of the left side of (27) as



$\xi \rightarrow \infty$ , shows that the integral function is a constant. Hence,

$$\Gamma(\xi) = \frac{C\sqrt{k-\xi}}{k_0^2 \epsilon_1 - \xi^2} \quad (28)$$

After taking the transforms of both sides of (21) with respect to  $x$ , substituting the value of  $\Gamma(\xi)$  from (28) and writing the inverse transform, one obtains the following expression

$$H_y(x, z) = \frac{iC}{4\pi} \int_{-\infty}^{\infty} \left( -\frac{\epsilon_2}{\epsilon_1} \xi \pm i\sqrt{k^2 - \xi^2} \right) \frac{e^{i\sqrt{k^2 - \xi^2}|z|} e^{i\xi x}}{(k_0^2 \epsilon_1 - \xi^2)\sqrt{k+\xi}} d\xi \quad (29)$$

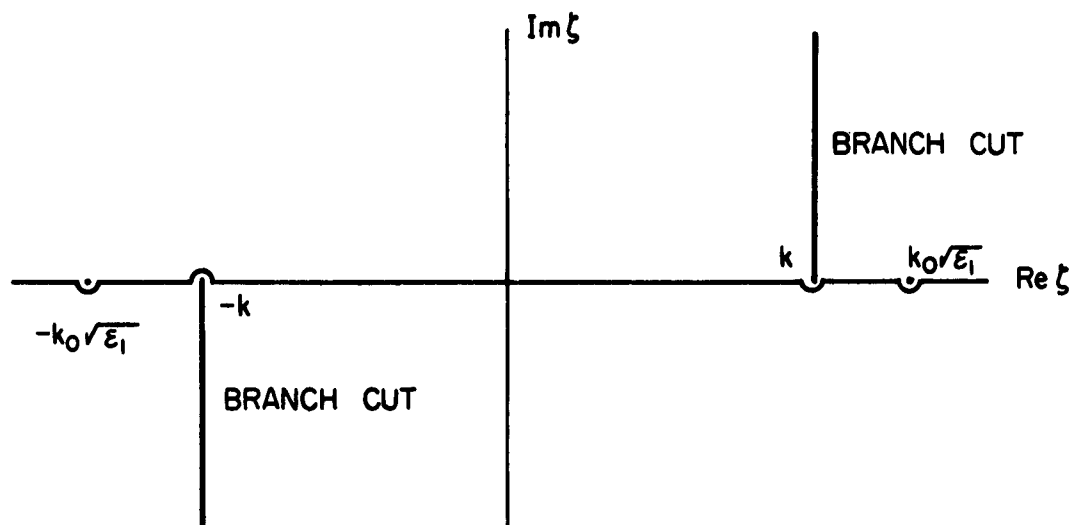
In (29), the upper and lower signs hold, respectively, for  $z$  positive or negative.

The contour for integration in (29) is shown in Fig. 4. For  $x > 0$ , the integral can be evaluated by closing the contour in the upper half-plane. For  $z > 0$ , the residue at the pole  $\xi = k_0 \sqrt{\epsilon_1}$  is seen to be zero, if the fact that  $\epsilon_2 < 0$  is noted. The contribution from the pole  $\xi = -k_0 \sqrt{\epsilon_1}$  is

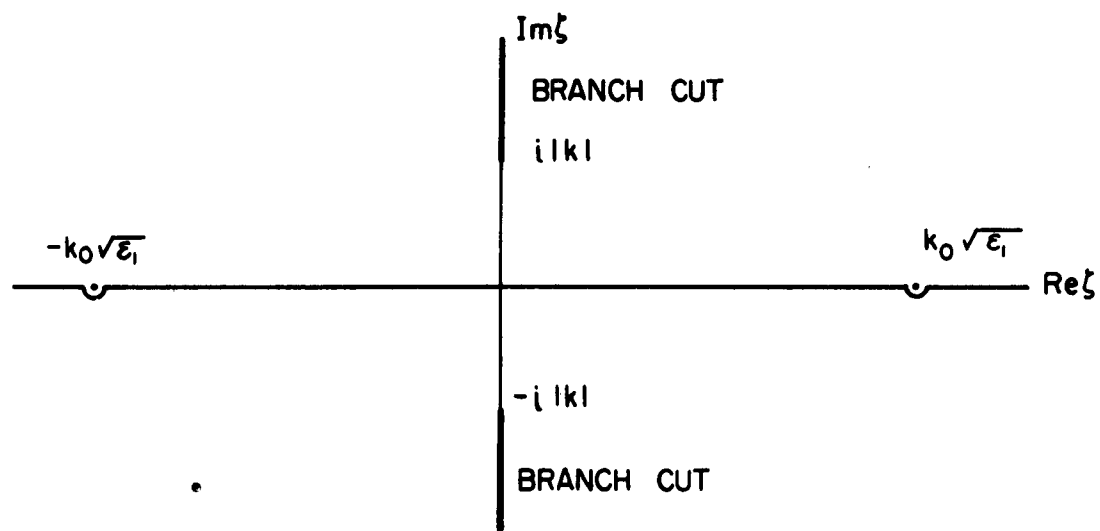
$$H_y^i(x, z) = \frac{C|\epsilon_2|}{2\epsilon_1 \sqrt{k - k_0 \sqrt{\epsilon_1}}} e^{-ik_0 \sqrt{\epsilon_1} x - k_0 \frac{|\epsilon_2|}{\sqrt{\epsilon_1}} z} \quad (30)$$

This gives just the incident surface wave (15). Hence, the constant  $C$  can be determined and it is given by

$$C = \frac{2H_s \epsilon_1 \sqrt{k - k_0 \sqrt{\epsilon_1}}}{|\epsilon_2|} \quad (31)$$



(a)  $k$  REAL



(b)  $k$  IMAGINARY

FIG. 4 INTEGRATION CONTOUR

For  $z < 0$ , the residue at the pole  $\xi = -k_0 \sqrt{\epsilon_1}$  is zero and the contribution from the pole  $\xi = k_0 \sqrt{\epsilon_1}$  shows the reflected surface wave to be

$$H_y^r(x, z) = H_s \Gamma e^{ik_0 \sqrt{\epsilon_1} x + \frac{k_0 |\epsilon_2| z}{\sqrt{\epsilon_1}}},$$

where the reflection coefficient  $\Gamma$  is given by

$$\Gamma = \frac{\sqrt{k - k_0 \sqrt{\epsilon_1}}}{\sqrt{k + k_0 \sqrt{\epsilon_1}}} \quad (33)$$

It is desired to calculate the incident power, the power in the reflected surface wave and the power transmitted as a space wave, all per unit width of the screen. The power in the incident surface wave per unit width of the screen is obtained from the relation

$$\begin{aligned} P_i &= \int_0^\infty -\hat{x} \cdot \frac{1}{2} \operatorname{Re} \vec{E}(x, z) \times \vec{H}^*(x, z) dz \\ &= \frac{1}{2} \operatorname{Re} \int_0^\infty E_z^i(x, z) H_y^{i*}(x, z) dz \end{aligned} \quad (34)$$

With the use of (15) in (6), it is found that

$$E_z^i(x, z) = \frac{k_0}{\omega \epsilon_0 \sqrt{\epsilon_1}} H_s e^{-ik_0 \sqrt{\epsilon_1} x - k_0 \frac{|\epsilon_2|}{\sqrt{\epsilon_1}} z} \quad (35)$$

Together with (15) and (35), (34) yields

$$P_i = \frac{|H_s|^2}{4\omega\epsilon_0|\epsilon_2|} \quad (36)$$

In a similar way the power transmitted in the reflected surface wave per unit width of the screen is

$$P_r = \frac{|H_s| |\Gamma|^2}{4\omega\epsilon_0|\epsilon_2|} \quad (37)$$

The reflection coefficient  $S$ , which is defined as the fraction of the incident power carried by the reflected surface wave, is therefore given by

$$S = \frac{P_r}{P_i} = |\Gamma|^2 = \frac{\epsilon_1 - \sqrt{\epsilon}}{\epsilon_1 + \sqrt{\epsilon}} \quad (38)$$

The power radiated in the form of a space wave can be evaluated in the following manner. For this purpose, it is convenient to introduce the polar coordinates

$$x = \rho \cos \theta \quad z = \rho \sin \theta \quad (39)$$

and the following transformation:

$$\xi = k \cos \tau \quad (40)$$

With (39) and (40), (29) reduces to

$$H_y(\rho, \theta) = -\frac{iCk^2}{4\pi} \int \left( -\frac{\epsilon_2}{\epsilon_1} \cos \tau + i \sin \tau \right) \frac{e^{ik\rho \cos(\tau-\theta)} \sin \tau d\tau}{[k_0^2 \epsilon_1 - k^2 \cos^2 \tau][k + k \cos \tau]^{1/2}} \quad (41)$$

For  $k\rho \gg 1$ , (41) is evaluated by the saddle-point method. It yields the space wave part as follows:

$$H_y^R(\rho, \theta) = - \frac{iCk^2}{2\sqrt{2\pi k\rho}} \frac{\left(-\frac{\epsilon_2}{\epsilon_1} \cos \theta + i \sin \theta\right) \sin \theta}{(k_o^2 \epsilon_1 - k^2 \cos^2 \theta)(k + k \cos \theta)} e^{i(k\rho - \frac{\pi}{4})} \quad (42)$$

With (6) and (39), it may easily be shown that for  $k\rho \gg 1$ ,

$$E_\theta(\rho, \theta) \simeq - \frac{\epsilon_1 k}{\omega \epsilon_o \epsilon} H_y(\rho, \theta) \quad (43)$$

The outward power flow, per unit area, per unit length of the screen at an angle  $\theta$  is obtained from (42) and (43) to be

$$\begin{aligned} S^R &= \frac{1}{2} \operatorname{Re} \hat{\rho} \cdot \vec{E}(\rho, \theta) \times \vec{H}^*(\rho, \theta) = \frac{k\epsilon_1}{2\omega\epsilon_o\epsilon} |H_y^*(\rho, \theta)|^2 \\ &= \frac{\epsilon |C|^2 F(\theta)}{16\pi \rho \omega k \epsilon_o \epsilon_1^3} \quad , \end{aligned} \quad (44)$$

where

$$F(\theta) = \frac{(1 - \cos \theta)}{\left[1 - \frac{\epsilon}{\epsilon_1} \cos^2 \theta\right]} \quad (45)$$

$F(\theta)$  given in (45) is defined to be the radiation pattern. The total power  $P^R$  radiated in the form of space waves is

$$P^R = \int_0^{2\pi} S^R \rho \, d\theta = \frac{\epsilon |C|^2}{16\pi \omega k \epsilon_0 \epsilon_1^3} \int_0^{2\pi} F(\theta) \, d\theta \quad (46)$$

It can be readily shown that for  $F(\theta)$  in (45)

$$\int_0^{2\pi} F(\theta) \, d\theta = \frac{2\pi \epsilon_1}{|\epsilon_2|} \quad (47)$$

The use of (31) and (47) in (46) yields, after some simplification,

$$P^R = \frac{|H_s|^2 \sqrt{\epsilon}}{2\omega \epsilon_0 |\epsilon_2| (\epsilon_1 + \sqrt{\epsilon})} \quad (48)$$

The transmission coefficient  $T$ , which is defined as the ratio of the power radiated as a space wave to the incident power, both per unit width of the screen, is therefore

$$T = \frac{2\sqrt{\epsilon}}{\epsilon_1 + \sqrt{\epsilon}} \quad (49)$$

It may be verified from (38) and (49) that  $S + T = 1$ , as it should.

All the above results are derived under the assumption that  $k$  is real. It is, therefore, pertinent to find out the ranges of  $\Omega$  for which  $k^2$  and hence,  $\frac{\epsilon}{\epsilon_1}$  is real and positive. From (4) and (7), it is clear that  $\frac{\epsilon}{\epsilon_1}$  is always real. It can be shown by simple manipulation that

$$\frac{\epsilon}{\epsilon_1} = \frac{(\Omega^2 - \Omega_1^2)(\Omega^2 - \Omega_3^2)}{\Omega^2 (\Omega^2 - \Omega_2^2)} \quad (50)$$

where

$$\Omega_1 = -\frac{R}{2} + \sqrt{\frac{R^2}{4} + 1}$$

$$\Omega_2 = \sqrt{1 + R^2}$$

and

$$\Omega_3 = \frac{R}{2} + \sqrt{\frac{R^2}{4} + 1} \quad (51)$$

Further, it is easily established that  $\Omega_1 < R < \Omega_2 < \Omega_3$ . From (50), it is obvious that  $\frac{\epsilon}{\epsilon_1} > 0$  only in the frequency ranges  $\Omega_1 < \Omega < \Omega_2$  and  $\Omega_3 < \Omega < \infty$  (Fig. 2). Hence, the expressions for the reflection coefficient, the radiation pattern and the transmission coefficient given, respectively, in (38), (45) and (49) are valid only for  $\Omega_1 < \Omega < \Omega_2$  and  $\Omega_3 < \Omega < \infty$ .

The incident surface wave assumed in (15) is legitimate only for the frequency ranges defined in (19). Therefore,  $\Omega$  is also restricted to these ranges. Within the stipulated ranges of  $\Omega$ ,  $\frac{\epsilon}{\epsilon_1}$  and hence  $k^2$  is negative in the ranges  $0 < \Omega < \Omega_1$  and  $\Omega_2 < \Omega < \Omega_3$ . For  $k$  purely imaginary, the solution of (23) has to be modified, with the result in the final solution (28),  $k$  should be replaced by  $i|k|$ . Then it is obvious from (33) and (38) that

$$S = 1 \quad (52)$$

for  $0 < \Omega < \Omega_1$  and  $\Omega_2 < \Omega < \Omega_3$ . When  $k$  in (42) is replaced by  $i|k|$ , it is seen that the space wave is exponentially damped and hence, no power is radiated in the form of a space wave. Hence,

$$T = 0 \quad (53)$$

for  $0 < \Omega < \Omega_1$  and  $\Omega_2 < \Omega < \Omega_3$ . The power in the incident surface wave traveling on the top of the screen is totally reflected as a surface wave which travels on the bottom of the screen when  $\Omega$  is in the ranges:  $0 < \Omega < \Omega_1$  and  $\Omega_2 < \Omega < \Omega_3$ .

### Numerical Results

The reflection and the transmission coefficients are computed as a function of  $\Omega$  for a particular value of  $R$ , namely  $R = \sqrt{3}$ . It is seen from an examination of Fig. 5, that the reflection coefficient is unity from  $0 < \Omega < \Omega_1$ ; it falls off rapidly as  $\Omega$  is increased beyond  $\Omega_1$ , reaches a minimum and then increases to unity at  $\Omega = R$ . It starts again at unity when  $\Omega = \Omega_2$  and remains at that value for  $\Omega$  up to  $\Omega_3$ ; then it quickly falls to zero as  $\Omega$  is increased beyond  $\Omega_3$ . It is obvious that in the frequency ranges for which the reflected surface wave and the space wave are present, the major portion of the energy is transmitted as a space wave except for  $\Omega$  very near  $\Omega_1$ ,  $R$  and  $\Omega_3$ . For a certain frequency between  $\Omega_1$  and  $R$ , the transmission coefficient has a maximum, and in the frequency ranges considerably greater than  $\Omega_3$ , a negligible amount of incident power is reflected as a surface wave.

The radiation pattern  $F(\theta)$  given in (45) is plotted in Fig. 6 for  $R = \sqrt{3}$  and for three values of  $\Omega$ . It is found that the radiation pattern always has a null in the direction of the screen and a maximum in the opposite direction. It is found that for  $\Omega > \Omega_3$ , the maximum increases very rapidly with  $\Omega$ .



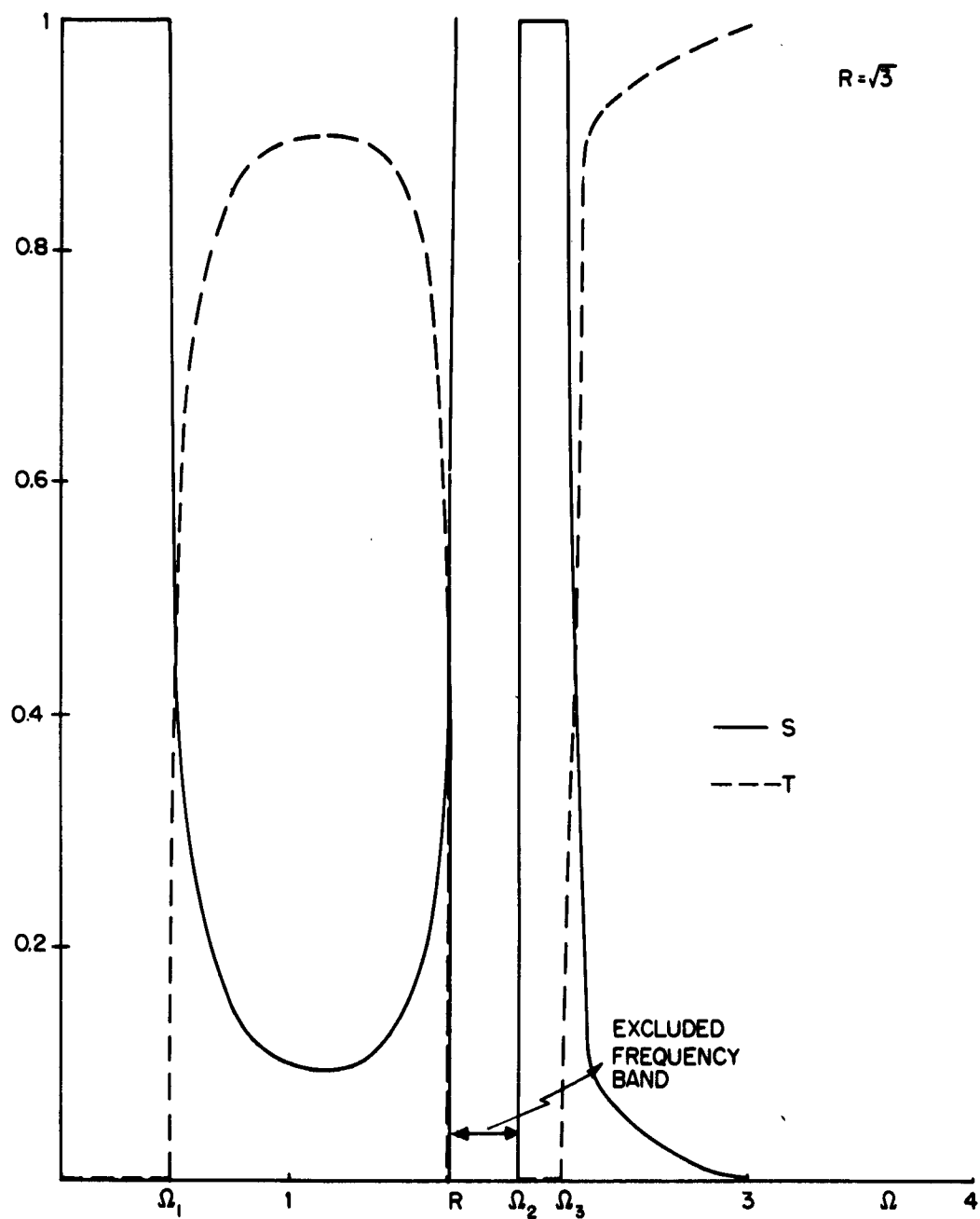


FIG. 5 REFLECTIONS AND TRANSMISSION COEFFICIENTS

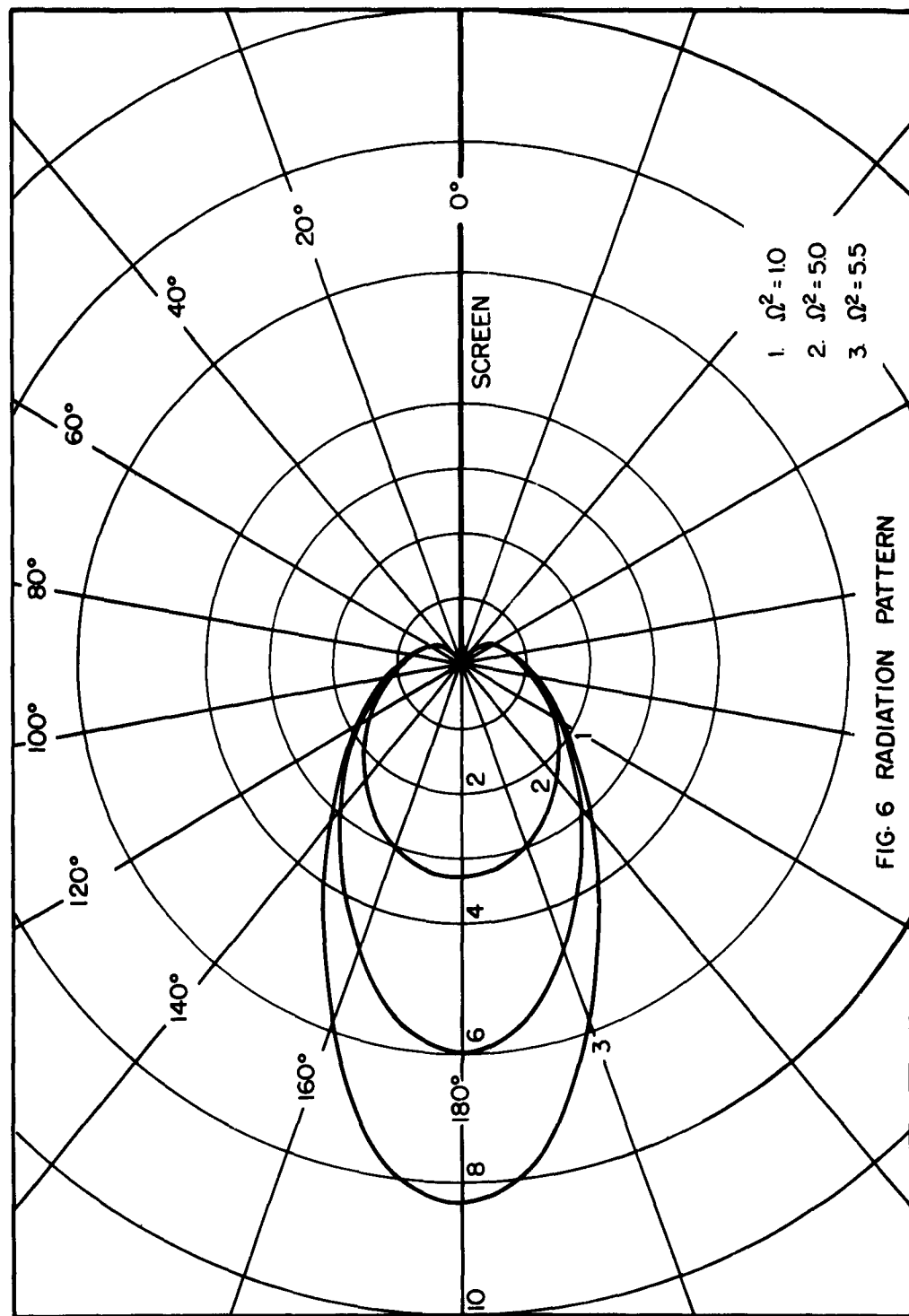


FIG-6 RADIATION PATTERN

as can be seen from the patterns for  $\Omega^2 = 5$  and  $\Omega^2 = 5 \cdot 5$ . For example, when  $\Omega = 3$ , the maximum value is nearly 20 times larger than that for  $\Omega = \sqrt{5 \cdot 5}$ . The reason for this rapid change in the maximum in the radiation pattern can be explained in the following manner. For  $\Omega > \Omega_3$ , the reflection coefficient  $S$  falls off very rapidly as  $\Omega$  is increased beyond  $\Omega_3$ . Hence, the major portion of the incident power is transmitted as a space wave. Also as  $\Omega$  is increased beyond  $\Omega_3$ ,  $\epsilon_2$  rapidly decreases to a very small value. As a consequence, the exponential attenuation in the incident plane wave is rapidly reduced. The incident wave becomes very nearly a homogeneous plane wave and therefore, the total incident power increases sharply when  $\Omega$  is increased further. Since the major portion of the incident power is transmitted as a space wave, the maximum in the radiation pattern rises sharply as  $\Omega$  is increased beyond  $\Omega_3$ .

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